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The Correlation Functions Near Intermittency in a One-Dimensional Piecewise Parabolic Map

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Piecewise parabolic maps constitute a family of maps in the fully developed chaotic state and depending on a parameter that can be smoothly tuned to a weakly intermittent situation. Approximate analytic expressions are derived for the corresponding correlation functions. These expressions produce power-law decay at intermittency and a crossover from power-law decay to exponential decay below intermittency. It is shown that the scaling functions and the exponent of the power law depend on the kind of the correlations.

KEY WORDS: Weak intermittency; chaos; phase transition; correlation function; scaling function; crossover behavior; critical slowing down.

INTRODUCTION

Intermittency is a fascinating phenomenon for two reasons. The first is the obvious change between chaotic and laminar or at least less chaotic behavior: In simple models this happens if in the region of interest there is no stable attractor and at the same time there exists a marginally unstable fixed point[(1)] or a marginally unstable attractor (crisis-induced intermittency[(2,3)]) or an attractor becoming unstable due to noise or due to the driving of another (strange) attractor (on–off intermittency[(4,5)]). Under these circumstances the system will again and again come close to the marginally unstable fixed point (or attractor, etc.) and while staying there the system remains less chaotic. Second, there is a suggestive analogy between the transition to intermittency and phase transitions of higher order in statistical physics[(6)] or the percolation transition of percolating clusters. In fact when approaching intermittency the correlations no longer

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decay exponentially, but rather follow a power law. One observes crossover behavior and critical slowing down. This is not unexpected because the larger and larger time intervals in which the dynamical system behaves "laminar" correspond to larger and larger ordered clusters that are observed near thermodynamic phase transitions. Correspondingly, renormalization methods have been applied to the intermittent transition. In particular it was shown[7] for weak intermittency that at the intermittent transition the correlations decay with a power law depending on the cusp of the map and the kind of the marginally unstable fixed point.

In this paper we calculate the scaling properties of the correlation functions for the piecewise parabolic map[8-10] in the neighborhood of the intermittent transition. We will show that in complete analogy to phase transitions of statistical physics a correlation function $c_{12}(m)$ can be written as a product of a singular expression times a scaling function

$$c_{12}(m) \propto \frac{1}{m^{1+\mu}} S_\mu(m\varepsilon), \qquad -1 < \mu \leqslant 1 \tag{1}$$

where m is the number of iterations between the first and second measurements and ε corresponds to $T - T_c$ in thermodynamic phase transitions.[3] In particular at $\varepsilon = 0$ the transition to intermittency is reached. We will show furthermore that the scaling function S as well as the exponent μ depend on the type of correlations. But the relation $-1 < \mu \leqslant 1$ holds and all scaling functions are simply related to each other.

The procedure we have chosen to calculate the correlation functions is based on determination of the eigenfunctions of the Frobenius–Perron operator connected to the piecewise parabolic map. This procedure is possible for the intermittent cases where the invariant measure remains smooth. The eigenfunctions and eigenvalues have been determined elsewhere[11,12] and the results are reported briefly in Section 2. Using these results in Section 3, we can determine how the mth power of the Frobenius–Perron operator (m arbitrarily high) acts on analytic functions. The result is valid in the neighborhood of intermittency and appropriate for calculating the correlation functions. Thus we can predict power-law decay at intermittency, crossover from power-law decay to exponential decay below intermittency, and a dependence of the power on the properties of the correlations at 0. These predictions are compared with numerical results in Section 4. A conclusion ends the paper.

[3] For $\mu = 1$ a logarithmic term may be present as well; cf. Section 3.

2. THE MODEL

We will discuss the road to intermittency for the piecewise parabolic map

$$x' = f(r, x) = \frac{1}{2r} \{1 + r - [(1-r)^2 + 4r\,|1-2x|]^{1/2}\} \tag{2}$$

For $r = 0$ one obtains the tent map

$$x' = f(r = 0, x) = 1 - |1 - 2x| \tag{3}$$

and for $r = 1$ one obtains the weak intermittent map

$$x' = f(r, x) = 1 - |1 - 2x|^{1/2}, \qquad f'(r = 1, 0) = 1 \tag{4}$$

Weak intermittency is characterized by the existence of a smooth invariant measure which can be given analytically in the case of the piecewise parabolic map[8]

$$p(x) = r + 1 - 2rx \tag{5}$$

Note that Eq. (5) is valid for any r in [0, 1], connected to the fact that the transition to intermittency is achieved by maintaining fully developed chaos.[8] Consequently the correlation function

$$c_{12}(m) = \langle c_1(f^m(x))\, c_2(x)\rangle - \langle c_1\rangle\langle c_2\rangle \tag{6}$$

can be calculated for analytic $c_2(x)$ if the eigenfunctions of the Frobenius–Perron operator $\mathscr{L}$ are known in the space of functions analytic in a neighborhood of [0, 1].

The eigenfunctions of the Frobenius–Perron operator have been calculated elsewhere[11, 12] by use of two different methods. Here we report the result only: Near intermittency the eigenequation of $\mathscr{L}$

$$\lambda\varphi(x) = \mathscr{L}\varphi(x) = \sum_{x = f(r, y)} \frac{1}{|f'(r, y)|}\,\varphi(y) \tag{7}$$

becomes simpler because the second branch of the Frobenius–Perron operator $\mathscr{L}$ can be neglected in a good approximation. The approximate eigenvalue equation

$$\lambda\varphi(f(r, x)) = \frac{1}{|f'(r, x)|}\,\varphi(x) \tag{8}$$

is appropriate for eigenfunctions well localized near 0. Solutions of Eq. (8) are

$$\varphi_n(x) = \frac{1}{(x+\varepsilon/a)^2}\left(\frac{x}{x+\varepsilon/a}\right)^n \qquad \lambda_n = e^{-\varepsilon}\cdot e^{-n\varepsilon}, \qquad n=0, 1,... \tag{9}$$

with

$$f(x) = x + \varepsilon x + ax^2 + \cdots$$

It is worth emphasizing that Eq. (9) explicitly displays a universal behavior of the spectral properties near intermittency, as these are to a good approximation independent of the precise form of the weakly intermittent map, except for the specific values of the parameters ε and a. In the piecewise parabolic map (2) ε and a can be expressed as

$$\varepsilon = \frac{1-r}{1+r} \qquad a = \frac{4r}{(1+r)^3} \tag{10}$$

The set $\{\varphi_n\}$ is *complete* (see ref. 11 and below), but has the disadvantage that the invariant measure p is an eigenfunction [see Eq. (5)] with eigenvalue 1 but it is *not* included in this set. We can correct this[4] by introducing the set $\{\psi_m\}$,

$$\psi_{-1} = p \qquad \psi_n = \varphi_n - \beta_n p \qquad \text{for} \quad n \geq 0 \tag{11}$$

with

$$\beta_n = \int_0^1 \varphi_n(x)\,dx$$

$$\lambda_{-1} = 1$$

$$\lambda_n = e^{-\varepsilon}\cdot e^{-n\varepsilon} \qquad \text{for} \quad n \geq 0$$

[4] Note that the adjoint eigenfunction with eigenvalue 1 is 1 itself.

This set represents a good approximation for the eigenfunction near $r=1$, the intermittent transition point.[5]

3. CALCULATION OF THE CORRELATION FUNCTION

If a smooth invariant measure p exists, the correlation function can be written as

$$c_{12}(m)=\int_0^1 c_1(f^m(x))\, c_2(x)\, p(x)\, dx-\langle c_1\rangle\langle c_2\rangle \tag{12}$$

with

$$\langle c_i\rangle=\int_0^1 c_i(x)\, p(x)\, dx$$

Using the Frobenius–Perron operator, we may write instead

$$c_{12}(m)=\int_0^1 c_1(x)[\mathscr{L}^m c_2 p](x)\, dx-\langle c_1\rangle\langle c_2\rangle \tag{13}$$

expanding

$$d_2(x)=c_2(x)\, p(x) \tag{14}$$

into a series of the $\{\psi_n\}$, we can calculate

$$d_2(m, x)=\mathscr{L}^m d_2(x) \tag{15}$$

However, the set $\{\varphi_n\}$ is much easier to handle than the set $\{\psi_n\}$. Fortunately, we can use $\{\varphi_n\}$ instead of $\{\psi_n\}$, replacing c_i by

$$\bar{c}_i(x)=c_i(x)-\langle c_i\rangle \tag{16}$$

To see this, we note first that in an expansion of d_2 the eigenfunction ψ_{-1} is not needed, since $\langle\bar{c}_2\rangle=0$. Second, since $\langle\bar{c}_1\rangle=0$, the correction terms $\beta_n p$ do not contribute to the correlation function $c_{12}(m)$. So from now on we will use $\bar{c}_i$ instead of c_i.

Next we perform a conjugation using the diffeomorphism

$$y=h(x)=\frac{x}{x+\varepsilon/a}\left(1+\frac{\varepsilon}{a}\right) \tag{17}$$

[5] Note that the $\{\psi_n\}$ also constitute a complete and independent set by construction.

In the new representations the approximate eigenfunctions $\{\varphi_n\}$ are just powers

$$\{\varphi_n\} \to \{y^n\} \tag{18}$$

which makes this set very convenient.[6] In the new representation we find

$$\hat{d}_2(m, y) = \mathscr{L}^m \hat{d}_2(y) \tag{19}$$

with

$$\hat{d}_2(m, y) = \frac{1}{h'(h^{-1}(y))} d_2(m, h^{-1}(y))$$

$$\hat{d}_2(y) = \frac{1}{h'(h^{-1}(y))} d_2(h^{-1}(y))$$

But $\hat{d}_2$ is analytic in the unit circle with respect to y; therefore it can be expanded into a power series

$$\hat{d}_2(y) = \sum_{\nu=0}^{\infty} a_\nu y^\nu$$

and

$$\mathscr{L}^m \hat{d}_2(y) = \sum_{\nu=0}^{\infty} a_\nu \mathscr{L}^m y^\nu$$

In a very good approximation

$$\mathscr{L}^m y^\nu = \lambda_\nu^m y^\nu = \lambda_0^m (\lambda_0^m y)^\nu, \qquad \lambda_0 = e^{-\varepsilon} < 1$$

Consequently,

$$\mathscr{L}^m \hat{d}_2(y) = \lambda_0^m \hat{d}_2(\lambda_0^m y) \tag{20}$$

or in the original system

$$\mathscr{L}^m d_2(x) = \lambda_0^m \frac{1}{[(1-\lambda_0^m)(a/\varepsilon)\, x + 1]^2} d_2\left(\lambda_0^m \frac{x}{(1-\lambda_0^m)(a/\varepsilon)\, x + 1}\right) \tag{21}$$

Note that the argument in d_2 remains small and only the behavior of $\bar{c}_2$ at 0 is of importance. In the following we assume $\bar{c}_2(0) \neq 0$. Then for $\varepsilon \ll 1$ the

[6] At the same time it is now obvious that the set $\{\varphi_n\}$ is complete.

expression $\mathscr{L}^m d_2(x)$ is strongly peaked around 0. Moreover, the formula fulfills the condition

$$\mathscr{L}^n \mathscr{L}^m d_2(x) = \mathscr{L}^{n+m} d_2(x) \tag{22}$$

The decrease of $c_{12}(m)$ depends now on $\bar{c}_1(x)$ in the neighborhood of 0. Let us consider the case[7]

$$\bar{c}_1(x) = x^\mu \qquad \text{for} \quad x \to 0, \quad \mu > -1 \tag{23}$$

Then, using Eqs. (13) and (21), we obtain

$$\int_0^1 \bar{c}_1(f^m(x))\, \bar{c}_2(x)\, p(x)\, dx$$

$$\approx \frac{\lambda_0^m}{[(1-\lambda_0^m)(a/\varepsilon)]^{\mu+1}} \int_0^{(1-\lambda_0^m)(a/\varepsilon)} \bar{c}_2(0)\, p(0) \frac{y^\mu}{(1+y)^2}\, dy$$

As long as $\mu < 1$ the upper limit of the integral does not play a role for large m and small ε. On the other hand, for $\mu \geqslant 1$ the upper limit is important. As a consequence, the correlation functions can be written in the asymptotic limit as a product of a power in m times a scaling function S_μ in the following way:

$$\begin{aligned} c_{12}(m) &\to \text{const} \cdot \frac{1}{m^2} S_1(m\varepsilon), && \mu > 1 \\ c_{12}(m) &\to \text{const} \cdot \frac{1}{m^{\mu+1}} S_\mu(m\varepsilon), && -1 < \mu < 1 \end{aligned} \tag{24}$$

Here S_μ is the scaling function is given by

$$S_\mu(z) = z^{\mu+1} \frac{e^{-z}}{(1-e^{-z})^{\mu+1}} \tag{25}$$

The case $\mu = 1$ is special. It contains a logarithmic term and cannot be expressed as a product of a scaling function and a function of m. At intermittency one obtains

$$c_{12}(m) \to \text{const} \cdot \frac{1}{m^2} \ln m \tag{26}$$

It follows from these equations that:

[7] $\langle c_1 \rangle$ is not defined for $\mu \leqslant -1$.

(i) For $m > 1/\varepsilon$ the correlation function $c_{12}(m)$ decays exponentially fast and for $m < 1/\varepsilon$, $c_{12}(m)$ decays as a power law indicating a *crossover behavior* and a critical slowing down when intermittency is reached.

(ii) At intermittency the decay of $c_{12}(m)$ cannot be faster than $1/m^2$. On the other hand, for $\mu \to -1$ it becomes extremely slow. The physical reason is that $\mu = -1$ corresponds to a nonintegrable singularity situated at the marginally unstable fixed point 0 of the intermittent map.

(iii) A decay faster than $1/m$ is only possible if $\bar{c}_1(0) = 0$ for $r = 1$.

Numerical examples are shown in the next section.

4. NUMERICAL RESULTS

We did numerical checks on Eq. (24) in a range of m values in which the correlation function itself changes by several orders of magnitude such that a small error in the scaling function or the power law would lead to strong disagreement between analytic formulas and the numerical results.

We begin with the case $\mu = 0$ by choosing first[8]

$$c_1(x) = c_2(x) = \begin{cases} 1 & \text{if} \quad x < B \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

For computing the correlation function analytically we use the asymptotic formula taking into account the finite integration limit. Then we obtain the analytic expression $c_{\mathrm{ana}}(m)$

$$c_{\mathrm{ana}}(m) = \frac{1}{m} S_0(m\varepsilon)\, a\bar{c}_1(0)\, \bar{c}_2(0)\, p(0) \int_0^{B(1-\lambda_0^m)(a/\varepsilon)} \frac{1}{y^2+1}\, dy \tag{28}$$

Numerically the correlation function $c_{12}(m)$ was calculated by performing 2×10^9 iterations for the cases[9] $r = 0.98$, 0.99, and 0.9999, setting $B = 5 \times 10^{-2}$. We plotted the ratio

$$R(m) = \frac{c_{12}(m)}{c_{\mathrm{ana}}(m)} \tag{29}$$

According to the results of the previous section [cf. Eq. (24)], $R(m, \varepsilon)$ should be approximately 1. This is verified in Fig. 1.

[8] In this section we choose for c_2 a step function because of its numerical advantages. Of course, the step function is not analytic. However, cutting the tail of its Fourier series, we have an analytic function arbitrarily close to a step function.

[9] For $r = 1$ the cusp of the map at $x = 1/2$ requires much higher precision in the numerics. Since there is nothing special in $r = 1$ compared to $r = 0.9999$, we chose the latter value.

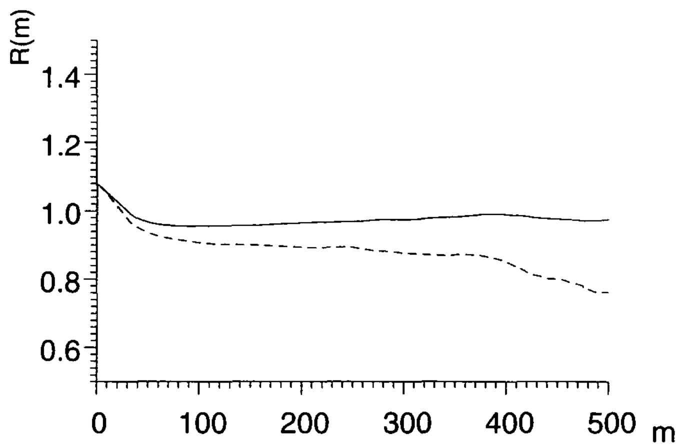

Fig. 1. The ratio between the numerically calculated correlation function and the analytic expression with c_1 and c_2 taken from Eq. (27). Solid line, $r = 0.9999$; dashed line, $r = 0.99$; dotted line, $r = 0.98$.

Next we checked Eq. (24) with $\mu = -1/2$ by choosing

$$c_1(x) = \begin{cases} \dfrac{1}{\sqrt{x}} & \text{if } \; x < B \\ 0 & \text{otherwise} \end{cases}$$
$$c_2(x) = \begin{cases} 1 & \text{if } \; x < B \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

For computing the correlation function $c_{\text{ana}}(m)$ we proceed as before and obtain

$$c_{\text{ana}}(m) = \frac{1}{\sqrt{m}} S_{-1/2}(m\varepsilon) \sqrt{a}\, \bar{c}_2(0)\, p(0) \int_0^{B(1-\lambda_0^m)(a/\varepsilon)} \frac{1}{\sqrt{y}\,(y^2+1)}\, dy \tag{31}$$

Numerically the correlation function $c_{12}(m)$ was calculated for $r = 0.98$, 0.99, and 0.9999, setting again $B = 5 \times 10^{-2}$ and doing 2×10^9 iterations. The results for the ratio $R(m)$ are shown in Fig. 2. We found that in particular for $r = 0.98$ the statistics became poor beyond $m = 400$, because this was beyond the crossover, and due to the exponential decay the value of the correlation function became extremely small. Nevertheless, below that value the numerical results clearly corroborate the analytic ones.

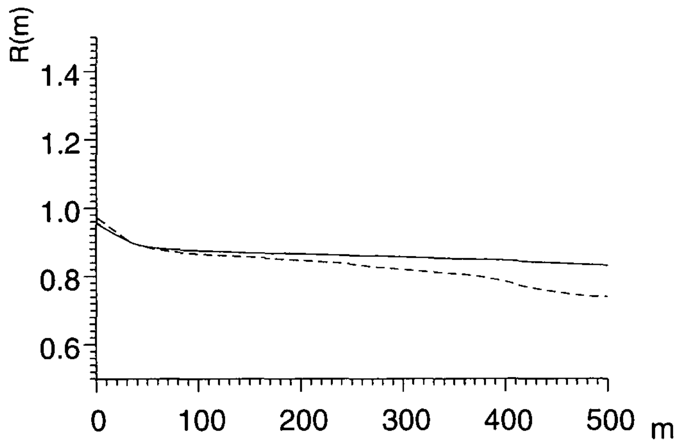

Fig. 2. The ratio between the numerically calculated correlation function and the analytic expression with c_1 and c_2 taken from Eq. (30). Solid line, $r = 0.9999$; dashed line, $r = 0.99$; dotted line, $r = 0.98$.

As a last example we present numerical checks on Eqs. (24) and (26) by choosing

$$\begin{aligned} c_1(x) &= \begin{cases} x - bx^2 & x < B \\ 0 & \text{otherwise} \end{cases} \\ c_2(x) &= \begin{cases} 1 & x < B \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{32}$$

Here b is determined such that

$$\langle x - bx_2 \rangle = 0 \qquad \text{in the intermittent case} \tag{33}$$

In the same manner as before we computed $c_{\text{ana}}(m)$ using $B = 5 \times 10^{-2}$. The results for the ratio $R(m)$ are shown in Fig. 3. Again for $r = 0.98$ the statistics is poor beyond $m = 300$, but below that value the results are remarkably good again.

One observes from the figures that $R(m) \to \text{const}$ is much better fulfilled than $R(m) \to 1$. This has a simple explanation: the approximate eigenfunctions corresponding to the upper edge of the spectrum are peaked near zero and the expansion coefficients of d_2 are sufficiently accurate only if d_2 is strongly concentrated around 0. Even if it is not, $\mathscr{L}^k d_2$ with increasing k has this property; therefore our method yields the correct asymptotics. Hence, when d_2 does not satisfy the above condition the initial deviations in the expansion coefficients will lead to $R(m) \to \text{const}$ rather than $R(m) \to 1$.

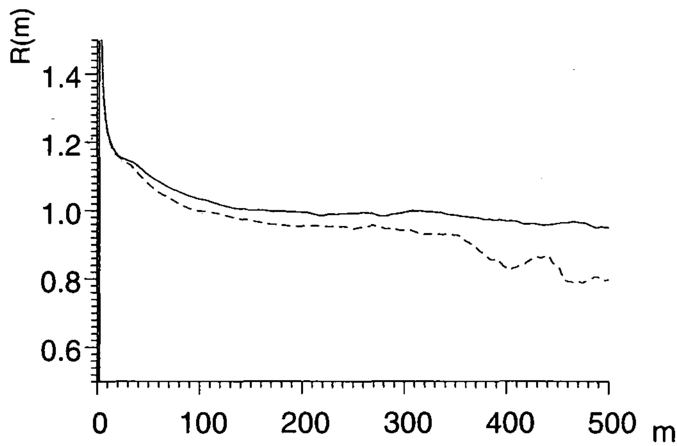

Fig. 3. The ratio between the numerically calculated correlation function and the analytic expression with c_1 and c_2 taken from Eq. (32). Solid line, $r = 0.9999$; dashed line, $r = 0.99$; dotted line, $r = 0.98$.

5. CONCLUSION

The piecewise parabolic map [cf. Eq. (2)] generates weak intermittency in the limit $r \to 1$. Exploiting the spectral properties of the corresponding Frobenius–Perron operator, we have derived analytic expressions for the correlation functions

$$c_{12}(m) = \langle c_1(f^m(x))\, c_2(x)\rangle - \langle c_1\rangle\langle c_2\rangle$$

which describes the asymptotics correctly. Assuming a c_2 with nonvanishing $c_2(0)$, a power-law decay is predicted at intermittency depending on the behavior of $c_1(x)$ at 0:

$$c_1(x) - \langle c_1\rangle \propto x^\mu \text{ for small } x\text{:} \quad c_{12}(m) \propto \frac{1}{m^{\mu+1}}, \quad \mu < 1 \tag{34}$$

$$c_1(x) - \langle c_1\rangle \propto x^\mu \text{ for small } x\text{:} \quad c_{12}(m) \propto \frac{1}{m^2}, \quad \mu > 1 \tag{35}$$

$$c_1(x) - \langle c_1\rangle \propto x \text{ for small } x\text{:} \quad c_{12}(m) \propto \frac{\ln m}{m^2} \tag{36}$$

Below intermittency the analytic expressions predict a crossover from power-law decay to exponential decay governed by scaling functions. All these predictions are confirmed quantitatively by the numerical computations.

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